

# 1234-avoiding permutations and Dyck paths

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**Abstract.** We define a map  $\nu$  between the symmetric group  $S_n$  and the set of pairs of Dyck paths of semilength  $n$ . We show that the map  $\nu$  is injective when restricted to the set of 1234-avoiding permutations and characterize the image of this map.

**Keywords:** restricted permutation, Dyck path.

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## 1 Introduction

We say that a permutation  $\sigma \in S_n$  *contains a pattern*  $\tau \in S_k$  if  $\sigma$  contains a subsequence that is order-isomorphic to  $\tau$ . Otherwise, we say that  $\sigma$  *avoids*  $\tau$ . Given a pattern  $\tau$ , denote by  $S_n(\tau)$  the set of permutations in  $S_n$  avoiding  $\tau$ .

The sets of permutations that avoid a single pattern  $\tau \in S_3$  have been completely determined in last decades. More precisely, it has been shown [10] that, for every  $\tau \in S_3$ , the cardinality of the set  $S_n(\tau)$  equals the  $n$ -th Catalan number, which is also the number of Dyck paths of semilength  $n$  (see e.g. [10]). Many bijections between  $S_n(\tau)$ ,  $\tau \in S_3$ , and the set of Dyck paths of semilength  $n$  have been described (see [4] for a fully detailed overview).

The case of patterns of length 4 appears much more complicated, due both to the fact that the patterns  $\tau \in S_4$  are not equidistributed on  $S_n$ , and the difficulty of describing bijections between  $S_n(\tau)$ ,  $\tau \in S_4$ , and some set of combinatorial objects.

In this paper we study the case  $\tau = 1234$ . An explicit formula for the cardinality of  $S_n(1234)$  has been computed by I. Gessel (see [2] and [5]), but there is no bijection (up to our knowledge) between  $S_n(1234)$  and some set of combinatorial objects.

We present a bijection between  $S_n(1234)$  and a set of pairs of Dyck paths of semilength  $n$ . More specifically, we define a map  $\nu$  from  $S_n$  to the set of pairs of Dyck paths, prove that every element in the image of  $\nu$  corresponds to a single element in  $S_n(1234)$ , and characterize the set of all pairs that belong to the image of the map  $\nu$ .

## 2 Dyck paths

A *Dyck path* of semilength  $n$  is a lattice path starting at  $(0, 0)$ , ending at  $(2n, 0)$ , and never going below the  $x$ -axis, consisting of up steps  $U = (1, 1)$  and down steps  $D = (1, -1)$ . A *return* of a Dyck path is a down step ending on the  $x$ -axis. A Dyck path is *irreducible* if it has only one return. An *irreducible component* of a Dyck path  $P$  is a maximal irreducible Dyck subpath of  $P$ .

A Dyck path  $P$  is specified by the lengths  $a_1, \dots, a_k$  of its ascents (namely, maximal sequences of consecutive up steps) and by the lengths  $d_1, \dots, d_k$  of its descents (maximal sequences of consecutive down steps), read from left to right. Set  $A_i = \sum_{j=1}^i a_j$  and  $D_i = \sum_{j=1}^i d_j$ . If  $n$  is the semilength of  $P$ , we have of course  $A_k = D_k = n$ , hence the Dyck path  $P$  is uniquely determined by the two sequences  $A = A_1, \dots, A_{k-1}$  and  $D = D_1, \dots, D_{k-1}$ . The pair  $(A, D)$  is called the *ascent-descent code* of the Dyck path  $P$ .

Obviously, a pair  $(A, D)$ , where  $A = A_1, \dots, A_{k-1}$  and  $D = D_1, \dots, D_{k-1}$ , is the ascent-descent code of some Dyck path of semilength  $n$  if and only if

- $0 < k \leq n - 1$ ;
- $1 \leq A_1 < A_2 < \dots < A_{k-1} \leq n - 1$ ;
- $1 \leq D_1 < D_2 < \dots < D_{k-1} \leq n - 1$ ;
- $A_i \geq D_i$  for every  $1 \leq i \leq k - 1$ .

It is easy to check that the returns of a Dyck path are in one-to-one correspondence with the indices  $1 \leq i \leq k$  such that  $A_i = D_i$ . Hence, a Dyck

path is irreducible whenever we have  $A_i > D_i$  for every  $1 \leq i \leq k - 1$ .

For example, the ascent-descent code of the Dyck path  $P$  in Figure 1 is  $(A, D)$ , where  $A = 3, 6$  and  $D = 2, 3$ . Note that  $A_1 > D_1$  and  $A_2 > D_2$ . In fact,  $P$  is irreducible.

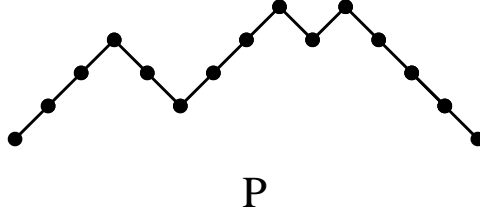


Figure 1

We describe an involution  $L$  due to Kreweras (a description of this bijection, originally defined in [7], can be found in [3]) and discussed by Lalanne (see [8] and [9]) on the set of Dyck paths. Given a Dyck path  $P$ , the path  $L(P)$  can be constructed as follows:

- if  $P$  is the empty path  $\epsilon$ , then  $L(P) = \epsilon$ ;
- otherwise:
  - flip the Dyck path  $P$  around the  $x$ -axis, obtaining a path  $E$ ;
  - draw northwest (respectively northeast) lines starting from the midpoint of each double descent (resp. ascent);
  - mark the intersection between the  $i$ -th northwest and  $i$ -th northeast line, for every  $i$ ;
  - $L(P)$  is the unique Dyck path that has valleys at the marked points (see Figure 2).

We define a further involution  $L'$  on the set of Dyck paths, which is a variation of the involution  $L$ , as follows:

- if  $P$  is the empty path  $\epsilon$ , then  $L(P) = \epsilon$ ;
- consider a Dyck path  $P$  and flip it with respect to a vertical line;

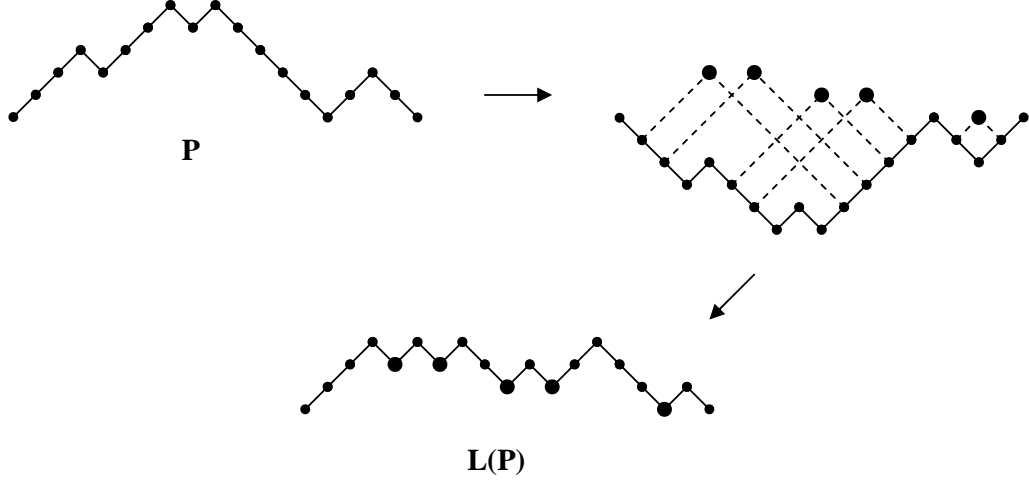


Figure 2. The map  $L$ .

- decompose the obtained path into its irreducible components  $U P_i D$ ;
- replace every component  $U P_i D$  with  $U L(P_i) D$  in order to get  $L'(P)$  (see Figure 3).

We point out that the map  $L'$  appears in a slightly modified version in the paper [3].

We now give a description of the map  $L'$  in terms of ascent-descent code. Obviously, it is sufficient to consider the case of an irreducible Dyck path  $P$ . Let  $(A, D)$  be the ascent-descent code of an irreducible path  $P$  of semilength  $n$ , with  $A = A_1, \dots, A_h$  and  $D = D_1, \dots, D_h$ . Straightforward arguments show that the ascent-descent code  $(A', D')$  of  $L'(P)$  can be described as follows:

- set  $\bar{A}_i = A_i - 1$  and set  $\hat{A} = [n-2] \setminus \{\bar{A}_1, \dots, \bar{A}_h\} = \{\hat{A}_1, \dots, \hat{A}_{n-2-h}\}$ , where the  $\hat{A}_i$ 's are written in decreasing order. Then,  $A'_i = n - \hat{A}_i$ .
- consider the set  $[n-2] \setminus \{D_1, \dots, D_h\} = \{\hat{D}_1, \dots, \hat{D}_{n-2-h}\}$ , where the  $\hat{D}_i$ 's are written in decreasing order. Then,  $D'_i = n - 1 - \hat{D}_i$ .

Finally, we introduce an order relation  $\leq$  on the set of Dyck paths of the same semilength. This order relation will be defined in three steps:

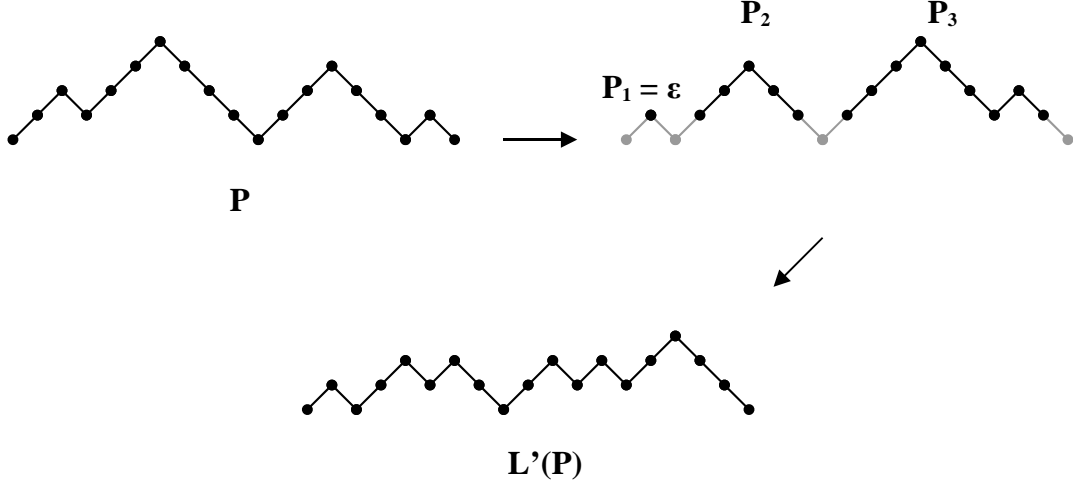


Figure 3. The map  $L'$ .

- Consider two irreducible Dyck paths  $P$  and  $Q$  of semilength  $n$ . Let  $(A, D)$  be the ascent-descent code of  $P$ , with  $A = A_1, \dots, A_k$  and  $D = D_1, \dots, D_k$ . We say that  $Q$  covers  $P$  in the relation  $\leq$  if the ascent code of  $Q$  is obtained by removing an integer  $A_i$  from  $A$  and the descent code of  $Q$  is obtained by removing an integer  $D_j$  for  $D$ , with  $j \geq i$ .

Roughly speaking,  $Q$  covers  $P$  if it can be obtained from  $P$  by “closing” the rectangles corresponding to an arbitrary collection of consecutive valleys of  $P$ ;

- the desired order relation  $\leq$  on the set of irreducible Dyck paths is the transitive closure of the above covering relation;
- the relation  $\leq$  is extended to the set of all Dyck path of a given semilength as follows: if  $P$  and  $Q$  are two arbitrary Dyck paths and  $P = P_1 P_2 \dots P_r$  and  $Q = Q_1 Q_2 \dots Q_s$  are their respective decompositions into irreducible parts, then  $P \leq Q$  whenever  $r = s$  and  $P_i \leq Q_i$  for every  $i$ .

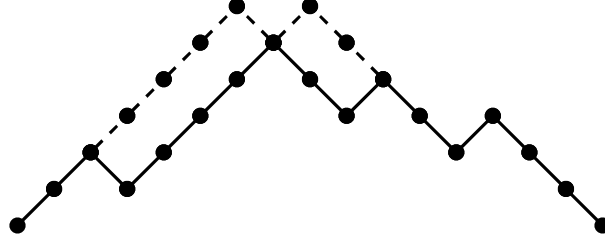


Figure 4. The dotted Dyck path covers the solid one.

### 3 LTR minima and RTL maxima of a permutation

Some of the well known bijections between  $S_n(\tau)$ ,  $\tau \in S_3$ , and the set of Dyck paths of semilength  $n$  (see [1], [6], and [10]) are based on the two notions of left-to-right minimum and right-to-left maximum of a permutation  $\sigma = x_1 x_2 \dots x_n$ :

- the value  $x_i$  is a *left-to-right minimum* (LTR minimum for short) at position  $i$  if  $x_i < x_j$  for every  $j < i$ ;
- the value  $x_i$  is a *right-to-left maximum* (RTL maximum) at position  $i$  if  $x_i > x_j$  for every  $j > i$ .

For example, the permutation

$$\sigma = 53482167$$

has the LTR minima 5, 3, 2, and 1 (at positions 1, 2, 5, and 6) and RTL maxima 7 and 8 (at positions 8 and 4).

We denote by  $vmin(\sigma)$  and  $pmin(\sigma)$  the sets of values and positions of the LTR minima of  $\sigma$ , respectively. Analogously,  $vmax(\sigma)$  and  $pmax(\sigma)$  denote the sets of values and positions of the RTL maxima of  $\sigma$ .

Recall that the *reverse-complement* of a permutation  $\sigma \in S_n$  is the permutation defined by

$$\sigma^{rc}(i) = n + 1 - \sigma(n + 1 - i).$$

For example, consider the permutation

$$\sigma = 247318956.$$

Then:

$$\sigma^{rc} = 451297368.$$

Note that the sets  $S_n(123)$  and  $S_n(1234)$  are closed under reverse-complement, namely,  $\sigma \in S_n(123)$  (respectively,  $\sigma \in S_n(1234)$ ) if and only if  $\sigma^{rc} \in S_n(123)$  (resp.  $\sigma^{rc} \in S_n(1234)$ ).

The first assertion in the next proposition goes back to the seminal paper [10], while the second one is an immediate consequence of the straightforward fact that  $x$  is a LTR minimum of a permutation  $\sigma$  at position  $i$  if and only if  $n + 1 - x$  is RTL maximum of  $\sigma^{rc}$  at position  $n + 1 - i$ :

**Theorem 1** *A permutation  $\sigma \in S_n(123)$  is completely determined by the two sets  $vmin(\sigma)$  and  $pmin(\sigma)$  of values and positions of its left-to-right minima. A permutation in  $S_n(123)$  is completely determined, as well, by the two sets  $vmax(\sigma)$  and  $pmax(\sigma)$  of values and positions of its right-to-left maxima.*

◇

Also 1234-avoiding permutations can be characterized in terms of LTR minima and RTL maxima.

This characterization can be found in [2] and is based on an equivalence relation on  $S_n$  defined as follows:  $\sigma \equiv \sigma' \iff \sigma$  and  $\sigma'$  share the values and the positions of LTR minima and RTL maxima.

For example,

$$1234 \equiv 1324.$$

Straightforward arguments lead to the following result stated in [2]:

**Theorem 2** *Every equivalence class of the relation  $\equiv$  contains exactly one 1234-avoiding permutation. In this permutation, the values that are neither LTR minima nor RTL maxima appear in decreasing order.*

◇

## 4 The maps $\lambda$ and $\mu$

We define two maps  $\lambda$  and  $\mu$  between  $S_n$  and the set  $\mathcal{D}_n$  of Dyck paths of semilength  $n$ . Given a permutation  $\sigma \in S_n$ , the path  $\lambda(\sigma)$  is constructed as follows:

- decompose  $\sigma$  as  $\sigma = m_1 w_1 m_2 w_2 \dots m_k w_k$ , where  $m_1, m_2, \dots, m_k$  are the left-to-right minima in  $\sigma$  and  $w_1, w_2, \dots, w_k$  are (possibly empty) words;
- set  $m_0 = n + 1$ ;
- read the permutation from left to right and translate any LTR minimum  $m_i$  ( $i > 0$ ) into  $m_{i-1} - m_i$  up steps and any subword  $w_i$  into  $l_i + 1$  down steps, where  $l_i$  denotes the number of elements in  $w_i$ .

The statement of Theorem 1 implies that the map  $\lambda$  is a bijection when restricted to  $S_n(123)$ .

Note that the ascent-descent code  $(A, D)$  of the path  $\lambda(\sigma)$  is obtained as follows:

- $A = n + 1 - m_1, n + 1 - m_2, \dots, n + 1 - m_{k-1}$ ;
- $D = p_2 - 1, p_3 - 1, \dots, p_k - 1$ , where  $p_i$  is the position of  $m_i$ .

We define a further map  $\mu : S_n \rightarrow \mathcal{D}_n$ :

- decompose  $\sigma$  as  $\sigma = u_h M_h u_{h-1} M_{h-1} \dots u_1 M_1$ , where  $M_1, M_2, \dots, M_h$  are the right-to-left maxima in  $\sigma$  and  $u_1, u_2, \dots, u_k$  are (possibly empty) words;
- set  $M_0 = 0$ ;
- associate with  $M_i$  ( $i > 0$ ) the steps  $U^{m_i - m_{i-1}} D$
- associate with each entry in  $u_i$  a  $D$  step.

Also in this case, the map  $\mu$  is a bijection when restricted to  $S_n(123)$ .

The ascent-descent code  $(A^*, D^*)$  of the path  $\mu(\sigma)$  is obtained as follows:

- $A^* = M_1, M_2, \dots, M_{h-1}$ ;



- $D^* = n - P_2, n - P_3, \dots, n - P_h$ , where  $P_i$  is the position of  $M_i$ .

In Figure 5 the two paths  $\lambda(\sigma)$  and  $\mu(\sigma)$  corresponding to  $\sigma = 6231754$  are shown.

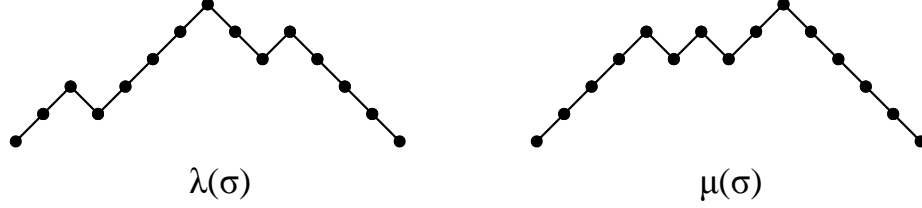


Figure 5. The Dyck paths corresponding to  $\sigma = 6231754$ .

We can now define a map  $\nu : S_n \rightarrow \mathcal{D}_n \times \mathcal{D}_n$ , setting

$$\nu(\sigma) = (\lambda(\sigma), \mu(\sigma)).$$

The statement of Theorem 2 implies that the map  $\nu$  is injective when restricted to  $S_n(1234)$ .

Note that the map  $\nu$  behaves properly with respect to the reverse-complement and the inversion operators:

**Proposition 3** *Let  $\sigma$  be a permutation in  $S_n$ . We have:*

- $\nu(\sigma) = (L, R) \iff \nu(\sigma^{rc}) = (R, L)$ , hence, the permutation  $\sigma$  is rc-invariant if and only if  $L = R$ .
- $\nu(\sigma) = (L, R) \iff \nu(\sigma^{-1}) = (\text{rev}(L), \text{rev}(R))$ , where  $\text{rev}(P)$  is the path obtained by flipping  $P$  with respect to a vertical line. Hence, the permutation  $\sigma$  is an involution if and only if both  $L$  and  $R$  are symmetric with respect to a vertical line.

◇

For example, consider  $\sigma = 6231754$ . The two paths associated with  $\sigma$  are shown in Figure 5. The permutation  $\sigma^{rc} = 4317562$  is associated with the two paths in Figure 6, while the permutation  $\sigma^{-1} = 4237615$  corresponds

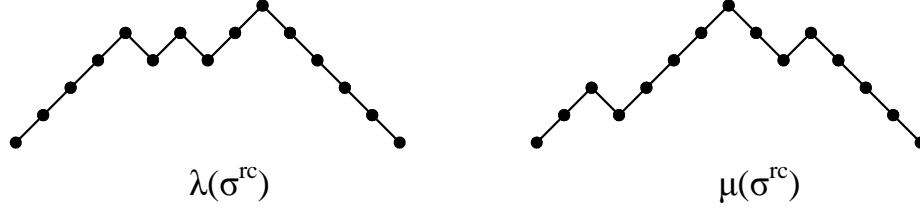


Figure 6. The Dyck paths corresponding to  $\sigma^{rc} = 4\ 3\ 1\ 7\ 5\ 6\ 2$ .

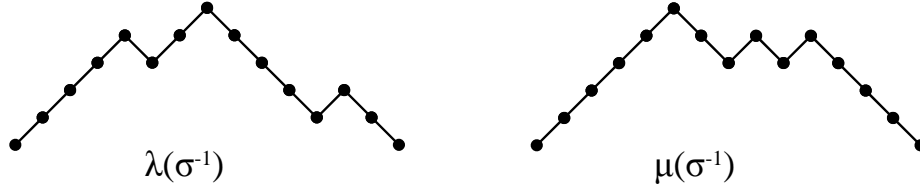


Figure 7. The Dyck paths corresponding to  $\sigma^{-1} = 4\ 2\ 3\ 7\ 6\ 1\ 5$ .

to the two paths in Figure 7.

Moreover, the map  $\nu$  has the following further property that will be crucial in the proof of our main result.

Recall that a permutation  $\sigma \in S_n$  is said *right-connected* if it does not have a suffix  $\sigma'$  of length  $k < n$ , that is a permutation of the symbols  $1, 2, \dots, k$ . For example, the permutation

$$\tau = 6\ 1\ 2\ 7\ 5\ 3\ 4\ 8$$

is right-connected, while

$$\sigma = 8\ 6\ 4\ 5\ 7\ 2\ 1\ 3$$

is not right-connected.

According to this definition, we can split every permutation into right-connected components:

$$\sigma = \mathbf{8\ 6\ 4\ 5\ 7\ 2\ 1\ 3}.$$

Note that, if a permutation  $\sigma$  is not right-connected,  $\sigma$  is the juxtaposition of a permutation  $\sigma''$  of the set  $\{t+1, \dots, n\}$  and the permutation  $\sigma'$  of the set  $\{1, \dots, t\}$ .

**Proposition 4** *Let  $\sigma$  be a non right-connected permutation in  $S_n$ , with  $\sigma = \sigma_1\sigma_2$ , where  $\sigma_1$  is a permutation of the set  $\{t+1, \dots, n\}$  and  $\sigma_2$  is a permutation of set of the set  $\{1, \dots, t\}$ . Then:*

$$\lambda(\sigma) = P_1P_2 \quad \mu(\sigma) = Q_1Q_2,$$

*with  $P_i = \lambda(\sigma_i)$  and  $Q_i = \mu(\sigma_i)$ ,  $i = 1, 2$ .*

◇

The order relation on Dyck paths defined in Section 2 can be exploited to define two order relations on the set  $S_n$  as follows:

- $\sigma \leq_\lambda \tau$  if and only if  $\lambda(\sigma) \leq \lambda(\tau)$ ;
- $\sigma \leq_\mu \tau$  if and only if  $\mu(\sigma) \leq \mu(\tau)$ .

These order relations can be intrinsically described as follows:

**Proposition 5** *Let  $\sigma, \tau \in S_n$ . We have  $\sigma \leq_\lambda \tau$  whenever:*

- $vmin(\tau) \subseteq vmin(\sigma)$ ;
- $pmin(\tau) \subseteq pmin(\sigma)$ ;
- *setting:*  
 $vmin(\sigma) = \{m_1, \dots, m_h\}$  (written in decreasing order),  
 $vmin(\sigma) \setminus vmin(\tau) = \{m_{i_1}, m_{i_2}, \dots, m_{i_r}\}$  (in decreasing order),  
 $pmin(\sigma) \setminus pmin(\tau) = \{p_{j_1}, p_{j_2}, \dots, p_{j_r}\}$  (in increasing order),  
then  $i_k < j_k$  for every  $k$ .

*Similarly,  $\sigma \leq_\mu \tau$  whenever:*

- $vmax(\tau) \subseteq vmax(\sigma)$ ;
- $pmax(\tau) \subseteq pmax(\sigma)$ ;
- *setting:*  
 $vmax(\sigma) = \{M_1, \dots, M_t\}$  (written in increasing order),  
 $vmax(\sigma) \setminus vmax(\tau) = \{M_{i_1}, M_{i_2}, \dots, M_{i_q}\}$  (in increasing order),  
 $pmax(\sigma) \setminus pmax(\tau) = \{P_{j_1}, P_{j_2}, \dots, P_{j_q}\}$  (in decreasing order),  
then  $i_k < j_k$  for every  $k$ .

◇

For example, consider the permutation

$$\sigma = 687325914.$$

We have  $vmin(\sigma) = \{6, 3, 2, 1\}$ ,  $pmin(\sigma) = \{1, 4, 5, 8\}$ ,  $vmax(\sigma) = \{4, 9\}$ , and  $pmax(\sigma) = \{9, 7\}$ . The permutation

$$\tau = 349268715$$

is such that  $vmin(\tau) = \{3, 2, 1\}$  and  $pmin(\tau) = \{1, 4, 8\}$ , hence,  $\sigma \leq_\lambda \tau$ . Moreover, the permutation

$$\rho = 271346589$$

is such that  $vmax(\rho) = \{9\}$  and  $pmax(\rho) = \{9\}$ , hence,  $\sigma \leq_\mu \rho$ .

## 5 Main results

We say that a pair of Dyck paths  $(P, Q)$  is *admissible* if there exists a permutation  $\alpha$  such that  $P = \lambda(\alpha)$  and  $Q = \mu(\alpha)$ . Needless to say, the set of admissible pairs is in bijection with the set of 1234-avoiding permutations.

We want to show that the operator  $L'$  on Dyck paths allows us to characterize the set of admissible pairs. We begin with a preliminary result concerning the pairs of Dyck paths corresponding to 123-avoiding permutations:

**Theorem 6** *For every  $\sigma \in S_n(123)$ , we have:*

$$\mu(\sigma) = L'(\lambda(\sigma)).$$

*Proof* Proposition 4, together with the definition of the map  $L'$ , allows us to restrict our attention to the right-connected case.

Recall (see [10]) that a permutation  $\sigma$  avoids 123 if and only if the set  $vmin(\sigma) \cup vmax(\sigma) = [n]$ . It is simple to check that, if  $\sigma$  is right-connected, the sets of LTR minima and RTL maxima are disjoint.

Consider now a permutation  $\sigma$  with LTR minima  $m_1, \dots, m_{k-1}, m_k = 1$  and RTL maxima  $M_1, \dots, M_{h-1}, M_h = n$ . Denote by  $(A, D)$  the ascent-descent code of the path  $P = \lambda(\sigma)$  and by  $(A^*, D^*)$  the ascent-descent code of the path  $\mu(\sigma)$ .

As noted before, the ascent code  $A'$  of  $L'(P)$  is obtained by computing the integers  $\bar{A}_i = A_i - 1$  and then considering the set  $\hat{A} = [n-2] \setminus \{\bar{A}_1, \dots, \bar{A}_{k-1}\}$ , which can be written as

$$\hat{A} = \{n - (n-1), n - (n-2), \dots, n-2\} \setminus \{n - m_1, \dots, n - m_{k-1}\}.$$

Since  $\{m_1, \dots, m_{k-1}\} \cup \{M_1, \dots, M_{h-1}\} = \{2, 3, \dots, n-1\}$ , we have

$$\hat{A} = \{n - M_1, \dots, n - M_{h-1}\}.$$

Hence,  $A' = A^*$ .

Similarly, the descent code  $D'$  of  $L'(P)$  is obtained by considering the set

$$\hat{D} = [n-2] \setminus \{D_1, \dots, D_{k-1}\} = [n-2] \setminus \{p_2 - 1, \dots, p_k - 1\}.$$

Since  $\{p_1, \dots, p_{k-1}\} \cup \{P_1, \dots, P_{h-1}\} = \{2, 3, \dots, n-1\}$ , we have

$$\hat{D} = \{P_2 - 1, \dots, P_{h-1} - 1\}.$$

Hence,  $D' = D^*$ .

◇

For example, the 123-avoiding permutation  $\sigma = 859762431$  corresponds to the pair of Dyck paths  $(P, L'(P))$  in Figure 3.

We are now in position to state our main result:

**Theorem 7** *A pair  $(P, Q)$  is admissible if and only if  $P \geq L'(Q)$  and  $Q \geq L'(P)$ .*

*Proof* Consider a permutation  $\sigma \in S_n(1234)$  and let  $\sigma'$  be the unique permutation in  $S_n(123)$  with the same LTR minima as  $\sigma$ , at the same positions. Obviously,  $\sigma' \leq_\mu \sigma$ , since in  $\sigma'$  every element that is not a LTR minimum is a RTL maximum (see Proposition 5). Recalling that  $\mu(\sigma') = L'(\lambda(\sigma)) = L'(P)$ , we get the first inequality. The other inequality follows from the fact that the pair  $(P, Q)$  is admissible whenever the pair  $(Q, P)$  is admissible.

Consider now a pair of Dyck paths  $(P, Q)$  such that  $P \geq L'(Q)$  and  $Q \geq L'(P)$ . Proposition 4 allows us to restrict to the case  $P, Q$  irreducible. Denote by  $\sigma$  and  $\tau$  the permutations in  $S_n(123)$  corresponding via  $\nu$  to the pairs  $(P, L'(P))$  and  $(L'(Q), Q)$ , respectively. Since  $P \geq L'(Q)$  and  $Q \geq L'(P)$ , we have  $\tau \leq_\lambda \sigma$  and  $\sigma \leq_\mu \tau$ .

We define a permutation  $\alpha \in S_n$  as follows:

- $\alpha(x) = \sigma(x)$  if  $x \in \text{pmin}(\sigma)$ ;
- $\alpha(x) = \tau(x)$  if  $x \in \text{pmax}(\tau)$ ;
- if  $x \notin \text{pmin}(\sigma) \cup \text{pmax}(\tau)$ , we have  $x \in \text{pmax}(\sigma) \setminus \text{pmax}(\tau) = \text{pmin}(\tau) \setminus \text{pmin}(\sigma) = \{p_{j_1}, \dots, p_{j_r}\}$ , written in increasing order. Set

$$\alpha(p_{j_k}) = m_{i_k},$$

where  $m_{i_1}, m_{i_2}, \dots, m_{i_r}$  are the elements in  $\text{vmin}(\tau) \setminus \text{vmin}(\sigma) = \text{vmax}(\sigma) \setminus \text{vmax}(\tau)$ , written in decreasing order.

The permutation  $\alpha$  is obtained as the concatenation of three decreasing sequences. Hence,  $\alpha$  avoids 1234. We have to prove that  $\text{vmin}(\alpha) = \text{vmin}(\sigma)$  and  $\text{vmax}(\alpha) = \text{vmax}(\tau)$ .

It is immediate that  $\text{vmin}(\sigma) \subseteq \text{vmin}(\alpha)$ . In order to prove that  $\text{vmin}(\sigma) = \text{vmin}(\alpha)$  it remains to show that the values  $m_{i_1}, m_{i_2}, \dots, m_{i_r}$  are not LTR minima of  $\alpha$ .

In fact, for every  $k$ , consider  $\alpha(p_{j_k}) = m_{i_k} = \tau(p_{i_k})$ . Consider the sets  $A = \{p_1, p_2, \dots, p_{i_k}\}$ ,  $B = \{m_1, m_2, \dots, m_{i_k}\}$ , and their subsets  $A' = \{p_{i_1}, p_{i_2}, \dots, p_{i_k}\}$  and  $B' = \{m_{i_1}, m_{i_2}, \dots, m_{i_k}\}$ . The  $k$  elements in  $B'$  do not belong to  $\text{vmin}(\sigma)$  (and hence, the  $i_k - k$  elements in  $B \setminus B'$  are the largest elements in  $\text{vmin}(\sigma)$ ). Proposition 5 ensures that each of them occupies in  $\alpha$  a position that is strictly greater than the position occupied in  $\tau$ . This implies that  $p_{j_k} < p_{i_k}$  and that at most  $k - 1$  elements in  $B'$  occupy in  $\tau$  a position that belongs to  $A$ . Hence, in  $\alpha$ , at least  $i_k - k + 1$  positions in  $A$  are occupied by entries belonging to  $\text{vmin}(\sigma)$ . This implies that there is in  $\alpha$  a position preceding  $p_{j_k}$  occupied by a value less than  $m_{i_k}$ . Hence,  $m_{i_k}$  is not a LTR minimum of  $\alpha$ .

Analogous arguments can be used to prove that  $\text{vmax}(\alpha) = \text{vmax}(\tau)$ . Hence,  $\nu(\alpha) = (P, Q)$ , as desired. ◇

For example, consider the pair of Dyck paths in Figure 8.

It can be checked that  $P \geq L'(Q)$  and  $Q \geq L'(P)$ . The permutations  $\sigma = \nu^{-1}((P, L'(P)))$  and  $\tau = \nu^{-1}((L'(Q), Q))$  are as follows:

$$\sigma = 498271653 \quad \tau = 759432816.$$

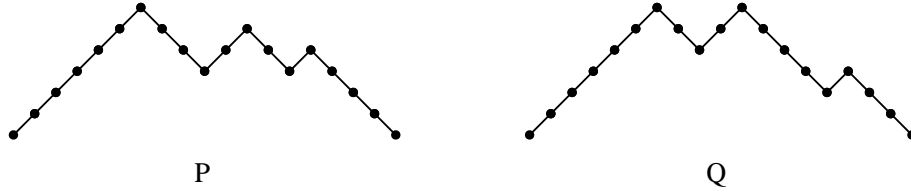


Figure 8

We have  $vmin(\sigma) = \{4, 2, 1\}$ ,  $pmin(\sigma) = \{1, 4, 6\}$ ,  $vmin(\tau) = \{7, 5, 4, 3, 2, 1\}$ ,  $pmin(\tau) = \{1, 2, 4, 5, 6, 8\}$ ,  $vmax(\sigma) = \{3, 5, 6, 7, 8, 9\}$ ,  $pmax(\sigma) = \{9, 8, 7, 5, 3, 2\}$ ,  $vmax(\tau) = \{6, 8, 9\}$ , and  $pmax(\tau) = \{9, 7, 3\}$ .

The permutation  $\alpha = \nu^{-1}((P, Q))$  is

$$\alpha = 479251836.$$

As expected,  $vmin(\alpha) = vmin(\sigma)$ ,  $pmin(\alpha) = pmin(\sigma)$ ,  $vmax(\alpha) = vmax(\tau)$ , and  $pmax(\alpha) = pmax(\tau)$ .

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